Preface

The subject of differential equations is very broad and compasses many aspects of analysis. This book provides a very simple and self-contained introduction to the spectral theory of second order elliptic differential operators which are involved in the most important equations in the physical sciences and engineering.

As a very rich field, spectral theory has been studied by different methods - Sturm-Liouville theory, Fredholm theory, separation of variables, Fourier and Laplace transforms, perturbation theory, eigenfunction expansions, etc. A reader of this book will find something of the underlying essence of all these developments.

The book is addressed to undergraduate and beginning graduate students of mathematics, physics and engineering who want to learn how the spectral theory elegantly helps to solve some mathematical problems that are related to the real world. The book differs from other introductory texts on spectral theory in that it is short, self-sufficient, contains clear and very simple proofs, and does not presuppose that the reader is familiar with multiple tools of the functional analysis.

The prerequisites are a very limited knowledge of some definitions concerning Hilbert and Banach space theory (up to Hahn-Banach theorem), a course on Lebesque integration, and a standard course on ordinary differential equations.

The topics of abstract functional analysis are presented insofar as they are vital for the applied spectral theory. In particular, the reader will find self-contained proofs of the results concerning linear functionals acting in Hilbert spaces, L_p and C, weak and strong compactness in L_p, Fredholm equations, the Fourier transform, weak partial derivatives, the Sobolev spaces, point spectrum, continuous spectrum and essential spectrum of linear operators, weak and singular solutions of differential equations, etc.

The symbiosis of the theory of compact operators in Hilbert spaces and the study of boundary-value problems is presented at two different levels: the one-dimensional case, and the three-dimensional case of spatial variables.

Applications include the spectrum of the operators generated by rotating fluids and by exponentially stratified fluids in the gravity field. The book contains some 25 examples and 60 exercises, most with detailed hints.

The book starts with the consideration of the compact sets in different functional spaces, including Banach spaces with basis, L_p and C, which is followed by the spectral decomposition of compact self-adjoint operators in Hilbert spaces (The Hilbert-Schmidt theorem). The first chapter is introductory and can be omitted by readers who are familiar with basic topics of functional analysis. Chapters 2 and 3 are devoted to the theory of compact operators in Hilbert spaces and the study of boundary-value problems for elliptic differential equations, stressing a deep effect of each on the other. Particularly, the Fredholm theorems are proved and details of the reduction of the Sturm-Liouville problem to an integral equation are discussed. Chapter 4 is dedicated to the integral representation of self-adjoint operators in the form of Stieltjes integral with respect to the spectral measure. Chapters 5 and 6 deal with the explicit spectral decomposition of the Laplacian in $L_2\left(R^3\right)$, and the study of the spectral structure of the Laplacian acting in the whole space. The importance of the applications to the Laplacian is evident, since Laplacian is involved in the mathematical description of every phenomena in isotropical and homogeneous medium. Chapter 7 studies eigenvalues of the Laplacian in bounded domains, making more precise and clear the co-existence of compact operators and boundary-value problems, whose consideration was started in Chapters 2 and 3. Chapter 8 is devoted to the spectrum of the operators generated by systems of different types of rotating and stratified fluid. The Weyl sequence is constructed, which enables not only the establishment of the spectral properties, but also allows for the explicit solution of a system of partial differential equations to be obtained.

I would like to express my sincere and profound gratitude to Prof. J.Lesmes, who read the manuscript and whose detailed remarks and suggestions helped to improve the text.

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Bogotá, November, 2004